

$$f) \sum \frac{n^3}{n!} z^n \quad \left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)^3}{n^3} \times \frac{1}{n+1} \rightarrow 0 \quad R = +\infty$$

$$g) \sum n^{(-1)^n} z^n \quad \left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)^{(-1)^{n+1}}}{n^{(-1)^n}} = \begin{cases} \frac{1}{n(n+1)} & n \text{ pair} \\ n(n+1) & n \text{ impair} \end{cases}$$

$\left| \frac{a_{n+1}}{a_n} \right|$ n'a pas de limite : d'Abel ne s'applique pas

$$a_n \leq n$$

$$a_n \geq \frac{1}{n}$$

$$\Rightarrow R = 1$$

$$\sum n z^n \text{ de rayon } 1 \Rightarrow R \geq 1$$

$$\sum \frac{1}{n} z^n \Rightarrow R \leq 1$$

$$z = e^{i\theta} \quad \sum n^{(-1)^n} e^{in\theta}$$

$$n^{(-1)^n} e^{in\theta} \not\rightarrow 0$$

$$\text{car } 2n^{(-1)^{2n}} e^{i2n\theta} \not\rightarrow 0$$

$\Rightarrow$ divergence sur le cercle d'unicité

$$h) \sum e^{i\omega(n)} z^n$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{e^{i\omega(n+1)}}{e^{i\omega(n)}} = \frac{e^{i\omega(n)} e^{i\omega(n)} - \sin(n) \sin(n)}{e^{i\omega(n)}}$$

$$\frac{1}{e} \leq e^{i\omega(n)} \leq e$$

$$\Rightarrow R = 1$$

$$e^{i\omega(n)} e^{in\theta} = e^{i\omega(n) + in\theta}$$

$$|e^{i\omega(n)} e^{in\theta}| \geq \frac{1}{e} \not\rightarrow 0$$

diverge sur le cercle d'unicité

$$i) \sum \min\left(\frac{1}{3^n}\right) z^n$$

$$\sum \min\left(\frac{1}{3^n}\right) z^n \quad \min\left(\frac{1}{3^n}\right) \sim \frac{1}{3^n} \Rightarrow R = 3$$

$$\Rightarrow \text{rayon } \sqrt[3]{3}$$

$$z = \sqrt[3]{3} e^{i\theta} : \min\left(\frac{1}{3^n}\right) e^{in\theta} \times 3^n \sim e^{in\theta} \Rightarrow \text{diverge sur } |z| = 1$$

$$d) \sum \frac{n!}{(2n)^n} z^n$$

$$\sum \frac{n!}{(2n)^n} z^n$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(2n+2)^{n+1}} \times \frac{(2n)^n}{n!} \rightarrow \frac{1}{2e}$$

$$= \underbrace{(n+1)}_{\downarrow \frac{1}{2}} \times \frac{1}{\underbrace{(2n+2)}_{\downarrow 2} \times \underbrace{\left(1+\frac{1}{n}\right)^n}_{\downarrow e}}$$

$$R = \sqrt[4]{2e}$$

$$z_0 = \sqrt[4]{2e} e^{i\theta}$$

$$\frac{n!}{(2n)^n} z_0^n = \frac{n!}{(2n)^n} (2e)^n \times e^{i2n\theta} = n! e^{i2n\theta} \not\rightarrow 0$$

donc donc que $n! |z| = 1$

$$h) \sum \sin(n) z^n$$

$$R \geq 1 \text{ car } |\sin(n)| \leq 1$$

$$\text{car: } n! \sin(n) \rightarrow 0 \quad \sin(n) z^n = \sin(n) \not\rightarrow 0 \quad \text{parce que!}$$

$$\sin(n) = \sin(n) \cos(0) + \cos(n) \sin(0) \rightarrow 0$$

$$\Rightarrow \sin(n) \cos(0) \rightarrow 0 \Rightarrow \cos(n) \rightarrow 0$$

$$\text{on } \sin(n)^2 + \cos(n)^2 = 1 \rightarrow 0 \quad \text{impossible!}$$

$$\sin(n) \not\rightarrow 0 \quad \text{donc donc que } n! |z| = 1 \quad \not\rightarrow 0$$

$$e) \sum_{n \geq 1} \ln(n) z^n$$

$$R = 1 \text{ (d'Abel)} \quad \text{d'Abel}$$

$$|\ln(n) e^{i\theta}| = |\ln(n)| \rightarrow +\infty$$

$\Rightarrow$ donc que $n! |z| = 1$

$$m) \sum \binom{2n}{n} z^n$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(2n)!}{(n!)^2} \times \frac{(n+1)!^2}{(2n+2)!} = \frac{(n+1)^2}{(2n+1)(2n+2)} \rightarrow \frac{1}{4}$$

$$\Rightarrow R = \frac{1}{4}$$

$$z_0 = \frac{1}{4} e^{i\theta}$$

$$\binom{2n}{n} z^n \sim \frac{(2n)!}{e^n} \sqrt{\frac{4}{\pi n}} \times \frac{e^{2n}}{n^{2n}} \times \frac{1}{2\pi n}$$

$$a_n z_0^n \sim \sqrt{\frac{4}{\pi n}} e^{i\theta} \not\rightarrow 0$$

$$\text{donc donc que } n! |z| = 1$$

Ex 2

$$a_0 > 0$$

$$a_{n+1} = \ln(1+a_n)$$

2) par récurrence immédiate $a_n > 0$

$$\text{car } a_n > 0 \Rightarrow \ln(1+a_n) > \ln(1) = 0$$

$$b) \quad \ln(1+x) \leq x \Rightarrow a_{n+1} = \ln(1+a_n) \leq a_n$$

c) TLN $\Rightarrow (a_n)$ converge vers l $l = \ln(1+l)$

$$\text{On } x > 0 \Rightarrow \ln(1+x) < x$$

$$\text{dnc } \underline{l=0}.$$

d) $\sum a_n 3^n$

$$a_n \rightarrow 0 \Rightarrow R \geq 1.$$

$$\frac{a_{n+1}}{a_n} = \frac{\ln(1+a_{n+1})}{a_{n+1}}$$

$$\text{or } \frac{\ln(1+x)}{x} \xrightarrow{x \rightarrow 0} 1$$

$$\Rightarrow R = 1$$

Exercice 4

$$\sum a_n z^n$$

rayon $R > 0$

$$\sum \frac{a_n}{n!} z^n$$

soit $z \in \mathbb{C}$ quelconque

z_0 tel $a_n z_0^n$ bornée

$$|z| > |z_0|$$

$$\frac{a_n z^n}{n!} =$$

$$\underbrace{a_n z_0^n}_{\text{bornée}} \times \underbrace{\left(\frac{z}{z_0}\right)^n \times \frac{1}{n!}}_{\text{bornée par CC}}$$

$\Rightarrow$ rayon $+\infty$.

exb

$$\begin{aligned}
 d) \sum_{n \geq 1} H_n x^n \quad H_n &= \sum_{k=1}^n \frac{1}{k} \xrightarrow{+ \infty} \\
 &= \left(\sum_{k=1}^{\infty} \frac{x^k}{k} \right) \times \sum_{n=1}^{\infty} x^n = -\ln(1-x) \times \frac{1}{1-x} \quad R=1 \\
 &= \frac{\ln(1-x)}{x-1} \quad R=1 \text{ per } \underline{\text{produit de Cauchy}} \text{ et } \sum H_n x^n \text{ diverge}
 \end{aligned}$$

$$\begin{aligned}
 c) \sum \frac{(n+1)(n-2)}{n!} x^n \\
 \left| \frac{a_{n+1}}{a_n} \right| &= \frac{(n+2)(n-1)}{(n+1)!} \times \frac{n!}{(n+1)(n-2)} = \frac{(n+2)(n-1)}{(n+1)(n+1)(n-2)} \rightarrow 0 \\
 R &= +\infty. \\
 &= \sum \frac{n^2 - n - 2}{n!} x^n \\
 &= \sum \frac{n(n-1)}{n!} x^n - 2 \sum \frac{x^n}{n!} \\
 &= n^2 \times \exp(x)'' - 2 \exp(x) \\
 &= (x^2 - 2) \exp(x).
 \end{aligned}$$

Exercice 7

$$\begin{aligned}
 &\sum \frac{\cos(n\theta)}{2^n n!} \\
 &= \operatorname{Re} \left(\sum \frac{e^{in\theta}}{2^n n!} \right) = \operatorname{Re} \left(\sum \frac{(e^{i\theta/2})^n}{n!} \right) = \operatorname{Re} \left(\exp(e^{i\theta/2}) \right) \\
 &= \operatorname{Re} \left(\exp\left(\cos\left(\frac{\theta}{2}\right) + i \sin\left(\frac{\theta}{2}\right)\right) \right) \\
 &= \operatorname{Re} \left(\exp\left(\cos\left(\frac{\theta}{2}\right)\right) \times e^{i \sin\left(\frac{\theta}{2}\right)} \right) \\
 &= \operatorname{Re} \left(\exp\left(\cos\left(\frac{\theta}{2}\right)\right) \times \left(\cos\left(\sin\left(\frac{\theta}{2}\right)\right) + i \sin\left(\sin\left(\frac{\theta}{2}\right)\right)\right) \right) \\
 &= \exp\left(\cos\left(\frac{\theta}{2}\right)\right) \times \cos\left(\sin\left(\frac{\theta}{2}\right)\right)
 \end{aligned}$$

exo 8 g)

$$f(u) = \frac{2u-1}{(u^2-u-2)^2} = \frac{u'}{u^2} = \left(-\frac{1}{u}\right)'$$

$$\Rightarrow \int f(u) du = -\frac{1}{u^2-u-2}$$

$$u^2-u-2 \quad \Delta = 1+8=9 \quad x_1, x_2 = \frac{1 \pm 3}{2} = -1 \text{ et } 2$$

$$u^2-u-2 = (u+1)(u-2)$$

$$\frac{1}{(u+1)(u-2)} = \frac{a}{(u+1)} + \frac{b}{(u-2)}$$

$$a = -\frac{1}{3} \quad b = \frac{1}{3}$$

$$= \frac{1}{3} \times \frac{1}{(u-2)} - \frac{1}{3} \times \frac{1}{(u+1)}$$

$$\frac{1}{u-2} = -\frac{1}{2} \times \frac{1}{1-\frac{u}{2}} = -\frac{1}{2} \times \sum_{k \geq 0} \left(\frac{u}{2}\right)^k$$

$$\frac{1}{1+u} = \sum_{k \geq 0} (-u)^k$$

$$\int f(u) du = -\frac{1}{6} \sum \left(\frac{u}{2}\right)^k - \frac{1}{3} \sum (-u)^k \quad R=1$$

$$= -\frac{1}{6} \times \sum \frac{u^k}{2^k} + 2u^k$$

$$= -\frac{1}{6} \times \sum_{k \geq 0} \frac{(2^{k+1}+1)}{2^k} u^k \quad R=1$$

$$\Rightarrow f(u) = -\frac{1}{6} \sum_{k \geq 1} k \frac{(2^{k+1}+1)}{2^k} u^{k-1}$$

$$f(u) = -\frac{1}{6} \sum_{k \geq 0} (k+1) \frac{(2^{k+2}+1)}{2^{k+1}} u^k \quad R=1.$$

Exercise 9

$$\forall z \in \mathbb{C} \quad c(z) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} z^{2n}$$

$$s(z) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$$

$$\cancel{c(z) \neq \sum_{n=0}^{+\infty} \frac{z^n}{n!}}$$

$$c(z) + i s(z) = \sum_{n=0}^{+\infty} \frac{z^n}{n!}$$

$$c(z) - i s(z) = \sum_{n=0}^{+\infty} \frac{(-z)^n}{n!}$$

$$c^2(z) + s^2(z) = (c(z) + i s(z)) (c(z) - i s(z))$$

$$= \sum_{n=0}^{+\infty} c_n z^n$$

$$c_n = \sum_{k=0}^n \frac{(-1)^k}{k!} \times \frac{1}{(n-k)!} = \frac{1}{n!} \times \sum_{k=0}^n \binom{n}{k} = \frac{1}{n!} \times 2^n$$

$$\Rightarrow c^2(z) + s^2(z) = \cancel{\sum_{n=0}^{+\infty} \frac{2^n}{n!} z^n} = \sum_{n=0}^{+\infty} c_n z^n = c_0 z^0 + 0 = 1$$

Kx 10

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n$$

$R > 0$

$$f(-z) = \sum_{n=0}^{+\infty} (-1)^n a_n z^n$$

$$\sum_{n=0}^{+\infty} a_{2n} z^{2n} = \frac{1}{2} (f(z) + f(-z))$$

Ex 12

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)(2n+2)} = \int_0^1 \text{Arctan } x \, dx$$

$$\text{Arctan}(x) = \sum_{n \geq 0} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$R=1.$$

$$\int_0^1 \text{Arctan}(x) \, dx = ?$$

CSSA $\Rightarrow$ il y a CUV sur $[0,1]$
car $|R_n(x)| \leq \frac{1}{2n+1} \rightarrow 0$

$$\begin{aligned} &= \sum_{n=0}^{+\infty} (-1)^n \int_0^1 \frac{x^{2n+1}}{2n+1} \, dx = \sum_{n=0}^{+\infty} (-1)^n \left[\frac{x^{2n+2}}{(2n+1)(2n+2)} \right]_0^1 \\ &= \sum_{n=0}^{+\infty} (-1)^n \frac{1}{(2n+1)(2n+2)} \end{aligned}$$

Ex 13

$$f(u) = \frac{\ln(u)}{u}$$

$$2) \ln(u) = \sum_{n \geq 0} (-1)^n \frac{u^{n+1}}{(n+1)!}$$

$$R = +\infty$$

$$\frac{\ln(u)}{u} = \sum_{n \geq 0} (-1)^n \frac{u^n}{(n+1)!}$$

$$\Rightarrow \infty \ln 1/2$$

$$b) f(u) = \frac{\ln(u)}{e^u - 1}$$

Ex 14

$$f: u \mapsto \frac{u}{1-u-u^2}$$

DSE.

$$(1-u-u^2)f(u) = u$$

$$f(u) = \sum a_n u^n$$

$$\Rightarrow \sum a_n u^n - \sum a_n u^{n+1} - \sum a_n u^{n+2} = u$$

$$\sum a_n x^n - \sum a_n x^{n+1} - \sum a_n x^{n+2} = x$$

$$= a_0 + a_1 x - a_0 x - \sum a_n x^{n+2} = x$$

$$+ \sum_{n \geq 2} a_n x^n - \sum_{n \geq 1} a_n x^{n+1}$$

$$= a_0 + (a_1 - a_0)x + \cancel{\sum_{n \geq 0} a_n x^{n+2}} - \cancel{\sum_{n \geq 1} a_n x^{n+1}}$$

$$+ \sum_{n \geq 2} a_n x^n - \sum_{n \geq 2} a_{n-1} x^n - \sum_{n \geq 2} a_{n-2} x^n = x$$

$$= a_0 + (a_1 - a_0)x + \sum_{n \geq 2} (a_n - a_{n-1} - a_{n-2})x^n = x$$

$$\Rightarrow a_0 = 0 \quad a_1 = 1$$

$$\forall n \geq 2 \quad a_n - a_{n-1} - a_{n-2} = 0$$

$$a_n = a_{n-1} + a_{n-2}$$

$$a_0 = 0 \quad a_1 = 1$$

$a_n =$ suite de Fibonacci