

Fiche d'exercice 14 : Calcul différentiel

Exercice 1

$f: I \rightarrow \mathbb{R}^n$ ne s'annule pas

$\forall t \in I$ $f(t)$ et $f'(t)$ sont colinéaires. ~~$\langle f(t), f'(t) \rangle = 0$~~

2) $u: t \mapsto \frac{1}{\|f(t)\|} \cdot f(t)$

$f \Delta^{\text{all}}, \| \cdot \| \Delta^{\text{all}} \Rightarrow u$ dérivable.
(car $\|v\| = \sqrt{\langle v, v \rangle}$ et $\langle \cdot, \cdot \rangle$ bilinéaire.)

$$\lambda: t \mapsto \frac{1}{\|f(t)\|} = \frac{1}{\sqrt{\langle f(t), f(t) \rangle}}$$

$$(\langle f(t), f(t) \rangle)' = 2 \langle f'(t), f(t) \rangle$$

$$x \mapsto \frac{1}{\sqrt{x}} \quad \left(\frac{1}{\sqrt{x}}\right)' = -\frac{1}{2\sqrt{x}}$$

$$\begin{aligned} \left(\frac{1}{\|f(t)\|}\right)' &= \cancel{\frac{-1}{2\|f(t)\|^2 \sqrt{\langle f(t), f(t) \rangle}}} - \frac{1}{2\langle f(t), f(t) \rangle^{3/2}} \times 2 \langle f'(t), f(t) \rangle \\ &= -\frac{\langle f'(t), f(t) \rangle}{\langle f(t), f(t) \rangle^{3/2}} \end{aligned}$$

$$u'(t) = -\frac{\langle f'(t), f(t) \rangle}{\langle f(t), f(t) \rangle^{3/2}} \cdot f(t) + \frac{1}{\|f(t)\|} \cdot f'(t)$$

$$f'(t) = \alpha f(t)$$

$$b) \quad u'(t) = \frac{1}{\|f(t)\|} \times \left[-\alpha \frac{\|f(t)\|^2}{\|f(t)\|^2} + \alpha \right] \cdot f(t) = 0$$

$$\Rightarrow u \text{ est constante sur } I$$

$$\Rightarrow \exists v \in \mathbb{R}^n \quad f(t) = \|f(t)\| \cdot v$$

$$\Rightarrow f(t) \in \text{Vect}(v)$$

f prend toutes les valeurs sur une droite vectorielle.

Exercice 2

$$f: (x, y) \mapsto \begin{cases} \frac{xy}{|x|+|y|} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

2) ? $\forall (x, y) \in \mathbb{R}^2 \quad |f(x, y)| \leq |y|$?

clair pour $(x, y) = (0, 0)$.

sin : $|f(x, y)| \leq |y|$

$$\Leftrightarrow |xy| \leq |x||y| + |y|^2$$

$$\Leftrightarrow |y|^2 \geq 0 \text{ vrai}$$

b) $\Rightarrow f(x, y) \xrightarrow{(0,0)} 0$

$$T_{(0,0)} f(x, 0) = \frac{f(x, 0) - f(0, 0)}{x} = 0 \rightarrow 0 = \frac{\partial f}{\partial x}(0, 0)$$

$$T_{(0,0)} f(0, y) = \frac{f(0, y) - f(0, 0)}{y} = 0 \rightarrow 0 = \frac{\partial f}{\partial y}(0, 0)$$

Exercice 3

$$f \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R}) \quad g: (u, v) \mapsto f(u^2 + v^2, uv)$$

$$(u, v) \mapsto u^2 + v^2 \text{ et } (u, v) \mapsto uv \text{ sont } \mathcal{C}^1$$

donc g est $\mathcal{C}^1$

Règle de la chaîne :

$$\frac{\partial g}{\partial x}(u, v) = \frac{\partial f}{\partial x}(u^2 + v^2, uv) \times 2u + \frac{\partial f}{\partial y}(u^2 + v^2, uv) \times v$$

$$\frac{\partial g}{\partial v}(u, v) = \frac{\partial f}{\partial x}(u^2 + v^2, uv) \times 2v + \frac{\partial f}{\partial y}(u^2 + v^2, uv) \times u$$

Exercice 4

$\varphi: \mathbb{R} \rightarrow \mathbb{R}$ dérivable $f: \mathbb{R}^* \times \mathbb{R} \rightarrow \mathbb{R}$
 $(x, y) \mapsto \varphi\left(\frac{y}{x}\right)$

Par composition f est dérivable sur $\mathbb{R}^* \times \mathbb{R}$

$$\frac{\partial f}{\partial x}(x, y) = \varphi'\left(\frac{y}{x}\right) \times \left(-\frac{y}{x^2}\right) \text{ et } \frac{\partial f}{\partial y}(x, y) = \varphi'\left(\frac{y}{x}\right) \times \frac{1}{x}$$

$$\Rightarrow x \frac{\partial f}{\partial x}(x, y) + y \frac{\partial f}{\partial y}(x, y) = -\frac{y}{x} \varphi'\left(\frac{y}{x}\right) + \frac{y}{x} \varphi'\left(\frac{y}{x}\right) = 0$$

Exercice 5

$n \in \mathbb{N}$ $f \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R})$ $\forall t \in \mathbb{R}, \forall (x, y) \in \mathbb{R}^2$
 $f(tx, ty) = t^n f(x, y)$

Soit

$$\varphi(x, y, t) = f(tx, ty) = t^n f(x, y)$$
$$\frac{\partial \varphi}{\partial t} = \frac{\partial f}{\partial x}(tx, ty) \times x + \frac{\partial f}{\partial y}(tx, ty) \times y = n t^{n-1} \times f(x, y)$$

En particulier pour $t=1$:

$$\frac{\partial f}{\partial x}(x, y) \times x + \frac{\partial f}{\partial y}(x, y) \times y = n \times f(x, y).$$

Exercice 6

$f: \mathbb{R}^2 \rightarrow \mathbb{R}$ de classe $\mathcal{C}^2$

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

$$F: (r, \theta) \mapsto f(r \cos \theta, r \sin \theta)$$

$$\frac{\partial F}{\partial \theta}(r, \theta) = \frac{\partial f}{\partial x}(r \cos \theta, r \sin \theta) \times (-r \sin \theta) + \frac{\partial f}{\partial y}(r \cos \theta, r \sin \theta) (r \cos \theta)$$

$$\frac{\partial F}{\partial r}(\theta) = \frac{\partial f}{\partial x}(r \cos \theta, r \sin \theta) \times \cos \theta + \frac{\partial f}{\partial y}(r \cos \theta, r \sin \theta) \times \sin \theta$$

$$\frac{\partial^2 F}{\partial \theta^2}(r, \theta) = \frac{\partial^2 f}{\partial x^2}(r \cos \theta, r \sin \theta) \times (-r \sin \theta)^2$$

$$+ \frac{\partial^2 f}{\partial y^2}(r \cos \theta, r \sin \theta) (-r \sin \theta) r \cos \theta$$

$$+ \frac{\partial^2 f}{\partial x^2}(r \cos \theta, r \sin \theta) \times (-r \cos \theta)$$

$$+ \frac{\partial^2 f}{\partial x \partial y}(r \cos \theta, r \sin \theta) \times (-r \sin \theta) (r \cos \theta)$$

$$+ \frac{\partial^2 f}{\partial y^2}(r \cos \theta, r \sin \theta) \times r^2 \cos^2 \theta$$

$$+ \frac{\partial^2 f}{\partial y^2}(r \cos \theta, r \sin \theta) (-r \sin \theta)$$

$$\frac{\partial^2 F}{\partial r^2}(r, \theta) = \frac{\partial^2 f}{\partial x^2}(r \cos \theta, r \sin \theta) \times \cos^2 \theta$$

$$+ \frac{\partial^2 f}{\partial y \partial x}(r \cos \theta, r \sin \theta) \times \cos \theta \times \sin \theta$$

$$+ \frac{\partial^2 f}{\partial x \partial y}(r \cos \theta, r \sin \theta) \times \sin \theta \times \cos \theta$$

$$+ \frac{\partial^2 f}{\partial y^2}(r \cos \theta, r \sin \theta) \times \sin^2 \theta$$

$$\begin{aligned}
\Rightarrow \frac{\partial^2 F}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 F}{\partial \theta^2} + \frac{1}{r} \frac{\partial F}{\partial r} &= \omega^2 \theta \times \frac{\partial^2 f}{\partial u^2} + 2 \sin \theta \frac{\partial^2 f}{\partial u \partial y} \\
&+ \frac{\partial^2 f}{\partial y^2} \times \sin^2 \theta + \cancel{\frac{\partial^2 f}{\partial u^2} \sin^2 \theta} - 2 \sin \theta \frac{\partial^2 f}{\partial u \partial y} \\
&+ \frac{\partial^2 f}{\partial y^2} \times \omega^2 \theta + \frac{1}{r} \left[-\frac{\partial f}{\partial u} \omega \theta + \frac{\partial f}{\partial y} \sin \theta \right] \\
&+ \frac{1}{r} \left[\frac{\partial f}{\partial u} \times \omega \theta + \frac{\partial f}{\partial y} \times \sin \theta \right] \\
&= \frac{\partial^2 f}{\partial u^2} [\omega^2 \theta + \sin^2 \theta] + \frac{\partial^2 f}{\partial y^2} [\omega^2 \theta + \sin^2 \theta] \\
&= \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial y^2} = \Delta f
\end{aligned}$$

Exercise 1 $\sum a_n z^n$ rayon $R > 0$

$$f(x+iy) = \sum_{n=0}^{+\infty} a_n (x+iy)^n \quad \Delta \text{ alle sur } D(0, R)$$

$$\frac{\partial f}{\partial x}(x,y) = \sum_{n=1}^{+\infty} n a_n (x+iy)^{n-1} \quad \cancel{\frac{1}{2} \frac{1}{n!}}$$

$$\frac{\partial f}{\partial y}(x,y) = \sum_{n=1}^{+\infty} n a_n (x+iy)^{n-1} \times i$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \sum_{n=2}^{+\infty} n(n-1) a_n (x+iy)^{n-2}$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = \sum_{n=2}^{+\infty} n(n-1) (x+iy)^{n-2} \times (-1)$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y) = 0$$

Exercise 8

$$(E) : \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = f.$$

$$\begin{cases} u = x \\ v = y - x \end{cases} \Leftrightarrow \begin{cases} x = u \\ y = u + v \end{cases}$$

$$\tilde{f}(u, v) = f(x, y) = f(u, u+v)$$

$$\frac{\partial \tilde{f}}{\partial u}(u, v) = \frac{\partial f}{\partial x}(u, u+v) + \frac{\partial f}{\partial y}(u, u+v)$$

$$\frac{\partial \tilde{f}}{\partial v}(u, v) = \frac{\partial f}{\partial y}(u, u+v)$$

$$\Rightarrow \Leftrightarrow \frac{\partial \tilde{f}}{\partial u}(u, v) = \tilde{f}(u, v)$$

$$\Leftrightarrow \tilde{f}(u, v) = \exp(u) \times \phi(v) \quad \phi \text{ de classe } C^1$$

$$\Leftrightarrow f(x, y) = \phi(y-x) \times e^x. \quad \phi \in C^1.$$

Exercise 9

$$(E) \quad \frac{\partial^2 f}{\partial x^2} - 2 \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2} = 0$$

$$\begin{cases} u = x \\ v = x + y \end{cases}$$

$$\tilde{f}(u, v) = f(x, y) = f(u, v-u)$$

$$\frac{\partial \tilde{f}}{\partial u}(u, v) = \frac{\partial f}{\partial x}(u, v-u) + \frac{\partial f}{\partial y}(u, v-u) \times (-1)$$

$$\frac{\partial \tilde{f}}{\partial v}(u, v) = \frac{\partial f}{\partial x}(u, v-u) \times 0 + \frac{\partial f}{\partial y}(u, v-u) \times 1$$

$$\frac{\partial^2 \tilde{f}}{\partial u^2}(u, v) = \frac{\partial^2 f}{\partial x^2}(u, v-u) + \frac{\partial^2 f}{\partial y \partial x}(u, v-u) \times (-1)$$

$$- \frac{\partial^2 f}{\partial x \partial y}(u, v-u) + \frac{\partial^2 f}{\partial y^2}(u, v-u)$$

$$= \frac{\partial^2 f}{\partial x^2}(u, v-u) - 2 \frac{\partial^2 f}{\partial x \partial y}(u, v-u) + \frac{\partial^2 f}{\partial y^2}(u, v-u)$$

$$\Rightarrow (\text{E}) \Leftrightarrow \frac{\partial^2 f}{\partial u^2} = 0$$

$$\Leftrightarrow \tilde{f}(u, v) = au + b + \phi(v) \quad \phi \in \mathcal{C}^2$$

$$\Leftrightarrow f(x, y) = ax + b + \phi(x + y) \quad \phi \in \mathcal{C}^2.$$

Exercice 11

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$f(x, y) = f(r \cos \theta, r \sin \theta) \in \mathcal{C}^1$$

$$= \tilde{f}(r, \theta)$$

$$\frac{\partial \tilde{f}}{\partial r}(r, \theta) = \frac{\partial f}{\partial x} \times \cos \theta + \frac{\partial f}{\partial y} \times \sin \theta$$

$$u, v, \theta$$

$$\Rightarrow \theta \in]-\frac{\pi}{2}, \frac{\pi}{2}[$$

$$\frac{\partial \tilde{f}}{\partial \theta}(r, \theta) = \frac{\partial f}{\partial x} \times (-r \sin \theta) + \frac{\partial f}{\partial y} \times r \cos \theta.$$

$$r = \sqrt{x^2 + y^2} \text{ et } u > 0 \Rightarrow \theta = \arctan\left(\frac{y}{x}\right)$$

$$a) \quad x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 0 \Leftrightarrow r \times \frac{\partial \tilde{f}}{\partial r}(r, \theta) = 0$$

$$\Leftrightarrow \frac{\partial \tilde{f}}{\partial r}(r, \theta) = 0$$

$$\Leftrightarrow \tilde{f}(r, \theta) = \phi(\theta) \quad \phi \in \mathcal{C}^1.$$

$$\Leftrightarrow f(x, y) = \phi\left(\arctan \frac{y}{x}\right).$$

$$b) \quad x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = \sqrt{x^2 + y^2} \Leftrightarrow r \frac{\partial \tilde{f}}{\partial r}(r, \theta) = r$$

$$\Leftrightarrow \frac{\partial \tilde{f}}{\partial r}(r, \theta) = 1$$

$$\Leftrightarrow \tilde{f}(r, \theta) = r + \phi(\theta)$$

$$\Leftrightarrow f(x, y) = \sqrt{x^2 + y^2} + \phi\left(\arctan \frac{y}{x}\right).$$

$$c) \quad x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = \frac{x}{\sqrt{x^2+y^2}}$$

$$\Leftrightarrow r \frac{\partial \tilde{f}}{\partial r}(r, \theta) = \frac{r \cos \theta}{r} = \cos \theta$$

$$\Leftrightarrow \tilde{f}(r, \theta) = \frac{1}{2} r^2 \cos \theta + \phi(\theta)$$

$$\Leftrightarrow f(x, y) = \frac{1}{2} \ln(\sqrt{x^2+y^2}) \times \frac{x}{\sqrt{x^2+y^2}} + \arctan\left(\frac{y}{x}\right) \cdot \phi(\theta)$$

$$d) \quad y \frac{\partial f}{\partial x}(x, y) - x \frac{\partial f}{\partial y}(x, y) = f(x, y)$$

$$\Leftrightarrow -\frac{\partial \tilde{f}}{\partial \theta}(r, \theta) = \tilde{f}(r, \theta)$$

$$\Leftrightarrow \tilde{f}(r, \theta) = k(r) \times e^{-\theta}$$

$$\Leftrightarrow f(x, y) = h(\sqrt{x^2+y^2}) \times e^{-\arctan\left(\frac{y}{x}\right)}$$

Exercise 12

Extreme local:

$$2) \quad f(x, y) = x^3 + y^3 \quad \frac{\partial f}{\partial x} = 3x^2 \quad \frac{\partial f}{\partial y} = 3y^2$$

$$\Rightarrow \text{1 sel } p^h \text{ unique en } (0,0).$$

$$\frac{\partial^2 f}{\partial x^2} = 6x$$

$$\frac{\partial^2 f}{\partial y^2} = 6y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 0$$

$$H(f)(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

$$x > 0 \quad f(x, 0) > f(0, 0) > f(-x, 0) : \text{even extrem.}$$

b) $g(x,y) = (x-y)^2 + (x+y)^3$ $\mathbb{R}^2$ can polynomials

$$\frac{\partial f}{\partial x} = 2(x-y) + 3(x+y)^2 \quad \frac{\partial f}{\partial y} = -2(x-y) + 3(x+y)^2$$

Probanten : $\begin{cases} \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \end{cases} \Leftrightarrow \begin{cases} 2(x-y) + 3(x+y)^2 = 0 \\ -2(x-y) + 3(x+y)^2 = 0 \end{cases}$

$$\Leftrightarrow \begin{cases} x+y=0 \end{cases} \Leftrightarrow \begin{cases} y=-x \\ x=0 \end{cases} \Leftrightarrow (x,y) = (0,0)$$

$$\frac{\partial^2 f}{\partial x^2} = 2 + 6(x+y) \quad \frac{\partial^2 f}{\partial y^2} = 2 + 6(x+y)$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = -2 + 6(x+y) \quad (\text{can } f \in \mathbb{R}^2, \text{ schwarz})$$

$$\Rightarrow Hf(0,0) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \quad \det = 0 \quad Hf(0,0) \text{ ist positiv} \\ t_n > 0$$

On ne peut rien en dire.

mais : pour $y=x$ $g(x,x) = 8x^3$ $\begin{cases} > 0 \text{ qd } x \rightarrow + \\ < 0 \text{ qd } x \rightarrow - \end{cases}$

$$\Rightarrow (0,0) \text{ n'est pas un extremum local (ni global.)}$$

Exercice 13

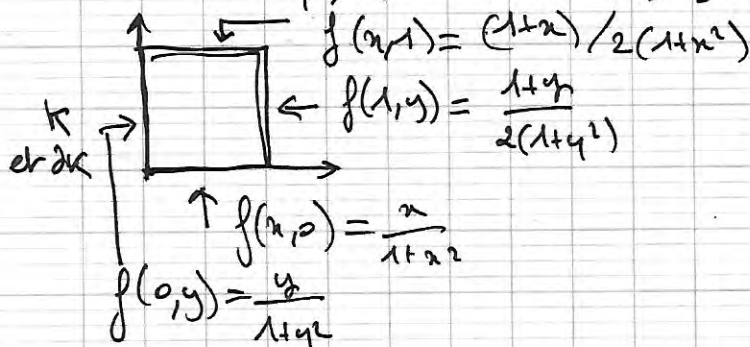
Maximum sur $K = [0,1]^2$ de $(x,y) \mapsto \frac{x+y}{(1+x^2)(1+y^2)}$

Existe d'après le TBA puisque f est continue ($f \neq$ rationnelle) sur $[0,1]^2$ fermé borné convexe ($[0,1]^2 = \overline{B((\frac{1}{2}, \frac{1}{2}), \frac{1}{2})}$ pour $\|\cdot\|_\infty$).

→ Si le maximum est atteint en l'intérieur $\Rightarrow$ c'est en un point critique.

Mais il peut être aussi atteint sur ∂K .

$$\partial K = \{0,1\} \times [0,1] \cup [0,1] \times \{0,1\}.$$



pts critiques: $\frac{\partial f}{\partial x} = \frac{1}{1+y^2} \times \frac{1-x^2-2xy}{(1+x^2)^2}$

$$\frac{\partial f}{\partial y} = \frac{1}{1+x^2} \times \frac{1-y^2-2xy}{(1+y^2)^2}$$

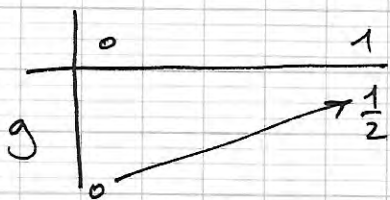
$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \Leftrightarrow \begin{cases} 1-x^2-2xy=0 \\ 1-y^2-2xy=0 \end{cases} \Leftrightarrow \begin{cases} x^2-y^2=0 \\ 1-3xy=0 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = \pm x \\ 1-3xy=0 \end{cases} \Leftrightarrow \begin{cases} y = x \\ 1-3x^2=0 \end{cases} \Leftrightarrow x=y=\frac{1}{\sqrt{3}}$$

Il y a un seul point critique $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$
En lequel la fonction prend la valeur:

$$f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{\frac{2}{\sqrt{3}}}{\left(\frac{4}{3}\right)^2} = \frac{3\sqrt{3}}{8}$$

Sur ∂K : $g(x) = \frac{x}{1+x^2}$ $g'(x) = \frac{(1-x)^2}{(1+x^2)^2} \geq 0$



$$h(x) = \frac{1+x}{2(1+x^2)} \quad h'(x) = \frac{1}{2} \frac{1+x^2 - 2x(1+x)}{(1+x^2)^2} = \frac{1}{2} \frac{1-x^2-2x}{(1+x^2)^2}$$

$h'(x) = 0 \Leftrightarrow 1-x^2-2x = 0$ $\Delta = 4+4=8 = (2\sqrt{2})^2$
 $x_1, x_2 = \frac{2 \pm 2\sqrt{2}}{-2} = \pm\sqrt{2} - 1$

$\Rightarrow$ $h(\sqrt{2}-1) = \frac{\sqrt{2}}{8-4\sqrt{2}} = \frac{1+\sqrt{2}}{4} \approx 0,6$

A graph of the function $h(x) = \frac{1+x}{2(1+x^2)}$ on the interval $[0, 1]$. The horizontal axis is labeled with 0, $\sqrt{2}-1$, and 1. The vertical axis is labeled with $\frac{1}{2}$. The curve starts at (0, $\frac{1}{2}$) and increases to the point ($\sqrt{2}-1$, $\frac{1}{2}$).

sur ∂K , la valeur maximale est atteinte en $(1, \sqrt{2}-1)$ et $(\sqrt{2}-1, 1)$ et a pour valeur $\frac{1+\sqrt{2}}{4}$.

On compare la valeur maximale sur ∂K et sur $\bar{K}$

$$\frac{1+\sqrt{2}}{4} \leq \frac{3\sqrt{3}}{8} \Leftrightarrow 8+8\sqrt{2} \leq 12\sqrt{3}$$

$$\Leftrightarrow 2+2\sqrt{2} \leq 3\sqrt{3}$$

$$\Leftrightarrow 4+8+8\sqrt{2} \leq 27$$

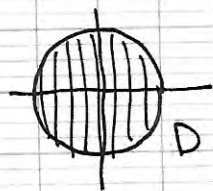
$$\Leftrightarrow 8\sqrt{2} \leq 15$$

$$\Leftrightarrow 2 \times 64 \leq 225 \Leftrightarrow 128 \leq 225 \text{ vrai}$$

$\Rightarrow$ le maximum est atteint au point intérieur $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ et a pour valeur maximale $f(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) = \frac{3\sqrt{3}}{8}$

Exercice 14

$$f: (x, y) \mapsto x^4 + y^4 - 2(x-y)^2 \text{ extrême sur } D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}$$



$D = \overline{B_{\|\cdot\|_2}((0,0), 2)}$ c'est la boule fermée de rayon 2 centrée en l'origine pour $\|\cdot\|_2$.

$\Rightarrow D$ est un fermé borné.

Puisque f est continue (car polynôme), f admet un min et un max sur D (TBA).

Si c'est en l'intérieur, c'est nécessairement un point critique :

$$\frac{\partial f}{\partial x} = 4x^3 - 4(x-y)$$

$$\frac{\partial f}{\partial y} = 4y^3 + 4(x-y)$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \Leftrightarrow \begin{cases} 4x^3 - 4(x-y) = 0 \\ 4y^3 + 4(x-y) = 0 \end{cases} \Leftrightarrow \begin{cases} x^3 + y^3 = 0 \\ \text{---} \end{cases}$$

$$\Leftrightarrow (x+y)(x^2 - xy + y^2) = 0 \Leftrightarrow \begin{cases} y = -x \text{ ou } x = y = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = -x \\ \text{ou} \\ 4x^3 - 8x = 0 \end{cases} \Leftrightarrow \begin{cases} y = -x \\ x(x^2 - 2) = 0 \end{cases} \Leftrightarrow \begin{matrix} (0,0) \\ (\pm\sqrt{2}, \mp\sqrt{2}) \end{matrix}$$

$\Rightarrow$ un seul point critique dans l'intérieur (car $(\pm\sqrt{2}, \mp\sqrt{2}) \notin D$)
 $(x, y) = (0, 0)$ au lequel $f(0, 0) = 0$.

Sur le bord: passage en coordonnées polaires

$$x = r \cos \theta \quad y = r \sin \theta$$

$$(r \cos \theta, r \sin \theta) \in \partial D \Leftrightarrow r = 2$$

$$f(x, y) = \tilde{f}(r, \theta); \quad \tilde{f}(2, \theta) = 16 \cos^4 \theta + 16 \sin^4 \theta - 8(\cos \theta - \sin \theta)^2$$

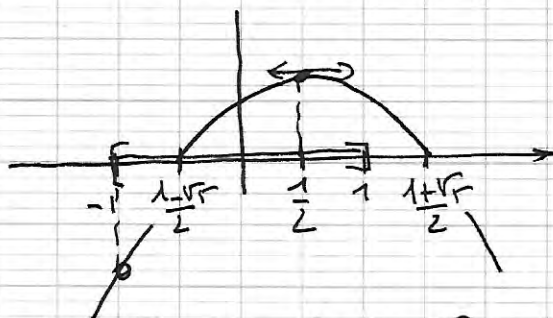
$$\begin{aligned} \tilde{f}(2, \theta) &= 16 \cos^4 \theta + 16 \sin^4 \theta - 8(\cos^2 \theta + \sin^2 \theta - 2 \sin \theta \cos \theta) \\ &= 8 \left[\underbrace{\cos^2 \theta (2 \cos^2 \theta - 1)}_{= \cos 2\theta} + \underbrace{\sin^2 \theta (2 \sin^2 \theta - 1)}_{= -\cos 2\theta} + \sin 2\theta \right] \end{aligned}$$

$$\begin{aligned}
 \tilde{f}(2, \theta) &= 8 \left[\cos(2\theta) (\cos^2 \theta - \sin^2 \theta) + \sin 2\theta \right] \\
 &= 8 \left[\cos(2\theta) \times \cos(2\theta) + \sin 2\theta \right] \\
 &= 8 \left[(2\cos^2 \theta - 1)(1 - 2\sin^2 \theta) + \sin 2\theta \right] \\
 &= 8 \left[2\cos^2 \theta + 2\sin^2 \theta - 4\cos^2 \theta \sin^2 \theta - 1 + \sin 2\theta \right] \\
 &= 8 \left[2 - 1 - (\sin 2\theta)^2 + \sin 2\theta \right] \\
 &= 8 \left[1 - \sin^2 2\theta + \sin 2\theta \right]
 \end{aligned}$$

On pose $X = \sin 2\theta$ $P = 1 + X - X^2$ $\Delta = 5$

$$X_{1,2} = \frac{1 \pm \sqrt{5}}{2}$$

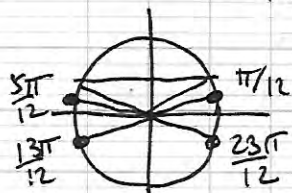
Ainsi $1 + X - X^2$ admet un maximum en $x = \frac{1}{2}$
 mais $\sin 2\theta \in [-1, 1]$: $\frac{1-\sqrt{5}}{2} < -1 < \frac{1}{2} < 1 < \frac{1+\sqrt{5}}{2}$



Donc sur $[-1, 1]$ $1 + X - X^2$ admet $\begin{cases} \text{un max en } x = \frac{1}{2} \\ \text{un min en } x = -1 \end{cases}$

$$\Rightarrow \tilde{f}(2, \theta) \text{ est maximal pour } \sin 2\theta = \frac{1}{2} \Leftrightarrow \begin{cases} 2\theta \equiv \frac{\pi}{6} [2\pi] \\ \text{ou} \\ 2\theta \equiv \frac{5\pi}{6} [2\pi] \end{cases}$$

$$\Leftrightarrow \begin{cases} \theta \equiv \frac{\pi}{12} [\pi] \\ \text{ou} \\ \theta \equiv \frac{5\pi}{12} [\pi] \end{cases} \Rightarrow \text{dans } [0, 2\pi] \quad \theta = \frac{\pi}{12} \text{ ou } \frac{5\pi}{12} \text{ ou } \frac{13\pi}{12} \text{ ou } \frac{23\pi}{12}$$



en bref $\tilde{f}(2, \theta) = 8 \times P\left(\frac{1}{2}\right) = 8 \times \frac{5}{4} = 10 > 0$

$\Rightarrow$ $\tilde{f}$ atteint son maximum en 4 points du bord :

$$(2\cos \frac{\pi}{12}, 2\sin \frac{\pi}{12}) \text{ et } (2\cos \frac{5\pi}{12}, 2\sin \frac{5\pi}{12}) \text{ et } (2\cos \frac{13\pi}{12}, 2\sin \frac{13\pi}{12}) \\ = (-2\cos \frac{\pi}{12}, 2\sin \frac{\pi}{12}) = -(2\cos \frac{\pi}{12}, 2\sin \frac{\pi}{12})$$

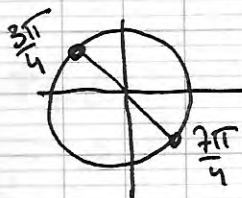
$$\text{et } (2\cos \frac{23\pi}{12}, 2\sin \frac{23\pi}{12}) = -(2\cos \frac{5\pi}{12}, 2\sin \frac{5\pi}{12}) \\ = (2\cos \frac{\pi}{12}, -2\sin \frac{\pi}{12})$$

4 max : $(\pm 2\cos \frac{\pi}{12}, \pm 2\sin \frac{\pi}{12})$

$f(2, \theta)$ est minimal sur ∂K pour $\sin 2\theta = -1$

$$\Leftrightarrow 2\theta \equiv 3\pi/2 \pmod{2\pi} \Leftrightarrow \theta \equiv 3\pi/4 \pmod{\pi}$$

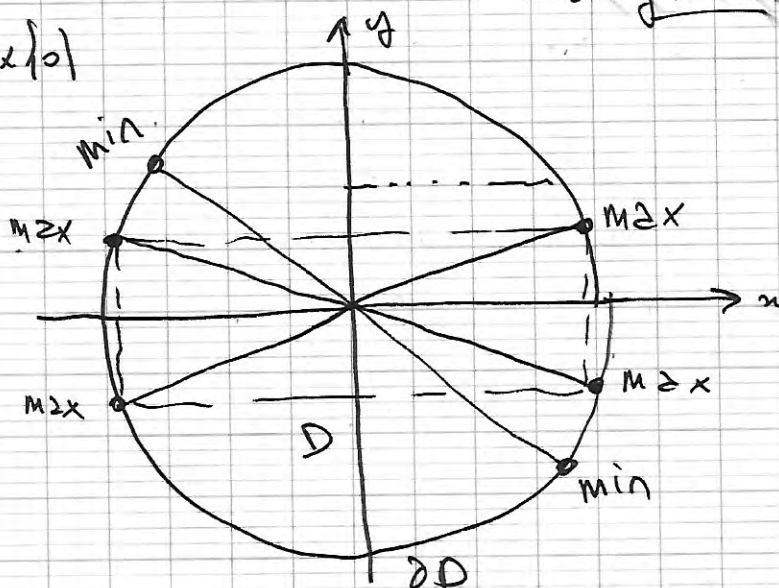
$\Rightarrow$ sur $[0, 2\pi]$ 2 minima atteints pour $\theta = 3\pi/4$ ou $7\pi/4$
pour lesquels :



$$\hat{f}(2, \theta) = 8 \times (-1) = -8 < 0$$

Donc f admet 2 minima, qui sont sur ∂D et égaux
à : $\pm(\sqrt{2}, -\sqrt{2})$ (qui peuvent être des points critiques
internes : il y a de bonnes chances
que ce soient en fait des mins
de f sur $\mathbb{R}^2$ (qui s'avère...))

$\mathbb{R}^2 \setminus \{0\}$



Exercice 15

$$f: (x, y) \mapsto xy(1-x-y)$$

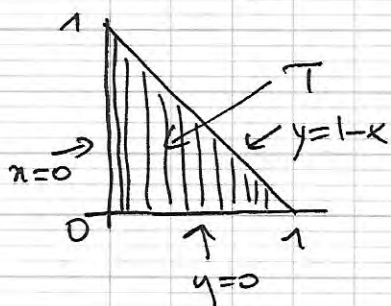
$$T = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0 \text{ et } y \geq 0 \text{ et } (x+y) \leq 1\}$$

$a: (x, y) \mapsto x$ $b: (x, y) \mapsto y$ $c: (x, y) \mapsto x+y$
sont continus car polynomiale.

$T = a^{-1}([0; +\infty[) \cap b^{-1}([0; +\infty[) \cap c^{-1}(]-\infty; 1])$
 $\Rightarrow T$ est un fermé car $\cap$ de préimages de fermés par des applications continues.

De plus T est borné: pour $\|\cdot\|_1$ $(x, y) \in T \Rightarrow \|(x, y)\|_1 \leq 1$
car $\|(x, y)\|_1 = |x| + |y| = x + y \leq 1$ pour $(x, y) \in T$.

En dim² presque tous les norms étant équivalents, T est fermé borné et donc par le f est continue (car polynomiale)
 f admet un maximum sur T
il peut être dans l'intérieur $\overset{\circ}{T}$ ou sur le bord ∂T .



sur ∂T $x=0$ ou $y=0$ ou $y=1-x$
 $\Rightarrow (x, y) \in \partial T \Rightarrow f(x, y) = 0$

f est nulle sur ∂T or $f(\frac{1}{4}, \frac{1}{4}) = \frac{1}{32} > 0$

Donc le maximum de f est atteint en son intérieur.

Cherchons les points critiques :

$$\frac{\partial f}{\partial x} = y(1-x-y) - xy = y(1-y-2x)$$

$$\frac{\partial f}{\partial y} = x(1-x-2y)$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \Leftrightarrow \begin{cases} y(1-y-2x) = 0 \\ \text{et} \\ x(1-x-2y) = 0 \end{cases} \quad (*)$$

$$n^{\circ} \quad x \neq 0 \text{ et } y \neq 0 \quad (*) \Leftrightarrow \begin{cases} 2x+y=1 \\ x+y=1 \end{cases} \Leftrightarrow x=y=\frac{1}{3}$$

$$n^{\circ} \quad x=y=0 \quad (n,y) \in \partial T$$

$$n^{\circ} \quad x=0 \text{ et } y \neq 0 \Rightarrow x=0 \text{ et } y=1 \Rightarrow (n,y) \in \partial T$$

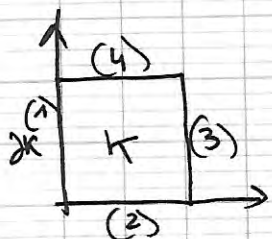
$$n^{\circ} \quad x \neq 0 \text{ et } y=0 \Rightarrow x=1 \text{ et } y=0 \Rightarrow (n,y) \in \partial T$$

Ainsi le maximum est atteint en $(\frac{1}{3}, \frac{1}{3})$ car c'est le seul point critique dans $\overset{\circ}{T}$.

$$f_{\max} = f\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{1}{27}$$

Exercice 1b

$K = [0, \frac{\pi}{2}]^2 \quad (x,y) \mapsto (\sin x)(\sin y)(\sin(x+y))$
 K est un fermé borné (boule fermée pour $\|\cdot\|_{\infty}$, centrée en $(\frac{\pi}{4}, \frac{\pi}{4})$ de rayon $\frac{\pi}{4}$).



sur ∂K :

$$(1): \quad x=0 \quad f(0,y) = 0$$

$$(2): \quad y=0 \quad f(x,0) = 0$$

$$(3): \quad x=\frac{\pi}{2} \quad f\left(\frac{\pi}{2}, y\right) = \sin(y) \sin\left(\frac{\pi}{2}+y\right) \\ = \sin(y) \times \cos(y) \\ = \frac{1}{2} \sin(2y)$$

max atteint en $x=\frac{\pi}{2} \quad y=\frac{\pi}{4}$ valeur: $\frac{1}{2}$

(4) idem: (par symétrie)

max atteint en $x=\frac{\pi}{4} \quad y=\frac{\pi}{2}$ valeur: $\frac{1}{2}$

Dans l'intérieur: le max est un point critique:

$$\frac{\partial f}{\partial x} = (\cos x)(\sin y)(\sin(x+y)) + (\sin x)(\sin y)(\cos(x+y)) \\ = \sin(y) [\cos(x) \sin(x+y) + \sin(x) \cos(x+y)] \\ = \sin(y) \times \sin(2x+y)$$

et par suite : $\frac{\partial f}{\partial y} = \sin(x) \times \sin(x+2y)$

$$\text{sur }]0, \frac{\pi}{2}[^2 : \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \Leftrightarrow \begin{cases} \sin(2x+y) = 0 \\ \text{ou} \\ \sin(x+2y) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} 2x+y \equiv 0 \pmod{\pi} \\ \text{ou} \\ x+2y \equiv 0 \pmod{\pi} \end{cases} \Leftrightarrow \begin{cases} 2x+y = \pi \\ \text{ou} \\ x+2y = \pi \end{cases} \Leftrightarrow x=y=\frac{\pi}{3}$$

Point critique : $(\frac{\pi}{3}, \frac{\pi}{3})$ $f(\frac{\pi}{3}, \frac{\pi}{3}) = \sin^2 \frac{\pi}{3} \times \sin \frac{2\pi}{3}$
 $= (\frac{\sqrt{3}}{2})^3 = \frac{3\sqrt{3}}{8} > \frac{1}{2}$

Ainsi le maximum est atteint au point $(\frac{\pi}{3}, \frac{\pi}{3})$ pour une valeur maximale de $\frac{3\sqrt{3}}{8}$

FIN